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Symmetries and Convergence of Normalizing Transformations

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1. INTRODUCTION

The theory of normal forms is one of the most useful tools in the investigation of the local behavior of an ordinary differential equation near a stationary point, cf. [1, 7] and the references therein. For differential equations with an analytic right hand side there is always a formal power series transformation into normal form, but frequently one encounters the phenomenon that no convergent transformation of this kind exists, cf. [1, 2]. (We use the abbreviation "convergent" for "convergent in an open neighborhood of 0.") Criteria for convergence and divergence are known, but there is still no general criterion to decide the convergence problem. In this note we prove such a criterion for the "interesting" two-dimensional systems (whose normal form is not trivial or almost trivial).

Thus consider a differential equation

$$\dot{x} = f^*(x) = B(x) + \sum_{j \geqslant 2} f_j^*(x) \tag{*}$$

in a neighborhood of 0 in $\mathbf{K}^2$($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$), where the right-hand side is analytic, B is a linear map, and f_j^* is a homogeneous polynomial of degree j for all $j \geqslant 2$. Furthermore we will restrict our attention to the case that B is semisimple and not zero and that the eigenvalues λ_1, λ_2 of B satisfy a

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relation $k_1\lambda_1 + k_2\lambda_2 = 0$, where k_1 and k_2 are nonnegative integers, $(k_1, k_2) \neq (0, 0)$, and k_1, k_2 are relatively prime. This characterizes the interesting case in dimension two where the normal form is not automatically a polynomial.

We denote by $[\cdot, \cdot]$ the usual *Lie bracket* of vector fields (thus $[g, h](x) = Dh(x)\, g(x) - Dg(x)\, h(x)$), and for a scalar-valued function φ the *Lie derivative* of φ with respect to g is defined by $L_g(\varphi)(x) := D\varphi(x)\, g(x)$.

It is known that (*) admits an invertible formal power series transformation into normal form, more precisely there are formal power series

$$H = \mathrm{Id} + \sum_{j \geqslant 2} H_j,$$

$$f = B + \sum_{j \geqslant 2} f_j$$

such that $DH(x)\, f^*(x) = f(H(x))$ and f is in normal form with respect to B, i.e., $[B, f] = 0$.

If there is a convergent power series H with this property then f is automatically convergent and it follows from $[B, f] = 0$ that there is an analytic one-parameter group $\{\exp(sB) : s \in \mathbf{K}\}$ of *symmetries* of $\dot{x} = f(x)$, i.e., an analytic one-parameter group of transformations which map parametrized solutions of $\dot{x} = f(x)$ to solutions again. Since $H = \mathrm{Id} + \cdots$ is locally invertible, we find that there is an analytic $g^* = B + \sum_{j \geqslant 2} g_j^*$ in a neighborhood of 0 such that $DH(x)\, g^*(x) = B(H(x))$ and $[f^*, g^*] = 0$, as the transformation respects Lie brackets. Thus g^* generates a local analytic one-parameter group of symmetries of $\dot{x} = f^*(x)$, the symmetries being obtained from the local flow of g^*. These symmetries are *nontrivial* (i.e., $g^* \notin \mathbf{K}f^*$) unless $f = B$, but in this case we may take $g = \mathrm{Id}$ and obtain nontrivial symmetries from $g^* = \mathrm{Id} + \cdots$.

Therefore, a necessary condition for the existence of a convergent transformation of (*) into normal form is the existence of a nontrivial local one-parameter group of symmetries (equivalently, the existence of an analytic $g^* \notin \mathbf{K}f^*$ in a neighborhood of 0 such that $[g^*, f^*] = 0$). Our main result is that this condition is also sufficient:

THEOREM. *The differential equation* (*) (*with B satisfying the conditions stated above*) *allows a convergent transformation into normal form if and only if it admits a nontrivial local one-parameter group of symmetries.*

2. PROOF OF THE THEOREM

We start by collecting a few facts. Let $f = B + \sum_{j \geqslant 2} f_j$ be a formal power series and $[B, f] = 0$. (Thus f is in formal normal form.) Then there is, up to a scalar constant, a unique homogeneous polynomial ρ such that $L_B(\rho)=0$, and for every formal power series φ such that $L_B(\varphi)=0$ there is a unique formal power series τ in one variable such that $\varphi=\tau(\rho)$. (If x_1, x_2 are the coordinates with respect to an eigenbasis (e_1, e_2) of B, then $\rho=x_1^{k_1}x_2^{k_2}$.)

LEMMA 1. *Let g be a formal power series. Then $[B, g]=0$ if and only if there are α_l, $\beta_l \in \mathbf{K}$ such that*

$$g=g_0+\sum_{l\geqslant 0}\rho^l(\alpha_l\,\mathrm{Id}+\beta_l B),$$

where $g_0=0$ for $\lambda_1\neq 0\neq\lambda_2$ and $g_0=\alpha_{-1}e_1$ for $\lambda_1=0$, $\lambda_2\neq 0$.

Proof. This follows from [1, Chap. II, Sect. 1.6], where

$$g(x)=\sum_{l\geqslant 0}\rho(x)^l\begin{pmatrix}\gamma_l x_1\\ \delta_l x_2\end{pmatrix}$$

(with respect to the eigencoordinates) is shown for $\lambda_1\neq 0\neq\lambda_2$. Since $\lambda_1\neq\lambda_2$ according to hypothesis, there are unique α_l, β_l such that $\alpha_l+\beta_l\lambda_1=\gamma_l$ and $\alpha_l+\beta_l\lambda_2=\delta_l$ for all l, and this finishes the proof. For $\lambda_1=0\neq\lambda_2$ note that $[B, g_0]=0$ is equivalent to $B(g_0)=0$; the remainder of the proof is completely analogous. ∎

If $g=B+\cdots$ and $\alpha_l=0$ for all $l>0$ then g satisfies Condition A of Theorem II in [2, Sect. 0]. This condition is important for the following reason: If $g^*=B+\sum_{j\geqslant 2}g_j^*$ is analytic near 0, and there is a formal power series $H=\mathrm{Id}+\cdots$ transforming g^* into normal form g such that g satisfies Condition A, then there is a convergent power series transforming g^* into (a convergent) normal form. This is the content of Theorem II just cited. (Note that the additional "Condition ω" on small divisors is automatically satisfied in our situation.) We will reduce the proof of the theorem to this case.

LEMMA 2. *Let $f=B+\sum_{j\geqslant 2}f_j$ be in formal normal form, φ a scalar-valued formal power series, and g a formal power series vector field. Then*

(i) $L_f(\varphi)=0\Rightarrow L_B(\varphi)=0$;

(ii) $[f, g]=0\Rightarrow[B, g]=0$.

Proof. Part (i) is a special case of [7, Proposition 1.8], and (ii) is proved in exactly the same manner. ∎

As usual, we call φ a (formal) *first integral* of f if $L_f(\varphi)=0$.

LEMMA 3. *Let $f = B + \sum_{j\geq 2} f_j$ be in formal normal form. Then f satisfies Condition A if and only if f has a nonconstant formal first integral.*

Proof. If f satisfies Condition A then $f = (1 + \sum_{l\geq 1} \beta_l \rho^l)B$, and $L_f(\rho) = 0$ follows from $L_B(\rho) = 0$.

Conversely, if ψ is a nonconstant first integral, then $\psi = \tau(\rho)$ for some nonconstant power series τ in one variable by Lemma 2, and $0 = L_f(\psi) = \tau'(\rho) \cdot L_f(\rho)$ shows $L_f(\rho) = 0$, as $\tau' \neq 0$.

Having $f = B + \sum_{l>0} \rho^l(\alpha_l \,\mathrm{Id} + \beta_l B)$ by Lemma 1, we get $L_f(\rho) = \sum_{l>0} \alpha_l(k_1 + k_2)\,\rho^{l+1}$ and therefore $\alpha_l = 0$ for all $l > 0$. ∎

LEMMA 4. *Let f be in formal normal form, f not satisfying Condition A, and g a formal power series vector field such that $g \notin \mathbf{K}f$ and $[f, g] = 0$. Then there are $c_1 \in \mathbf{K}^*$, $c_2 \in \mathbf{K}$ such that $g = c_1 B + c_2 f$.*

Proof. By Lemma 2 we have $[B, g] = 0$, thus $g = g_0 + \sum_{l\geq 0} \rho^l(\tilde{\alpha}_l \,\mathrm{Id} + \tilde{\beta}_l B)$ by Lemma 1. With $f = B + \sum_{l\geq 1} \rho^l(\alpha_l \,\mathrm{Id} + \beta_l B)$ we find $L_f(\rho) = \sum_{l\geq 1}(k_1 + k_2)\,\alpha_l\,\rho^{l+1} =: \varphi(\rho)$ and $L_g(\rho) = \tilde{\alpha}_{-1} + \sum_{l\geq 0}(k_1 + k_2)\,\tilde{\alpha}_l\,\rho^{l+1} =: \gamma(\rho)$ (with $\tilde{\alpha}_{-1} = 0$ for $\lambda_1 \neq 0 \neq \lambda_2$). Now $[f, g] = 0$ implies $[\varphi, \gamma] = 0$ and therefore $\gamma \in \mathbf{K}\varphi$ (since we are in dimension one and $\varphi \neq 0$ according to the hypothesis on f). Let $\gamma = c\varphi$ and $\tilde{g} := g - cf$. Then $[f, \tilde{g}] = 0$ and $L_{\tilde{g}}(\rho) = 0$, thus $\tilde{g} = (\sum_{l\geq 0} \rho^l \beta_l^*)\,B =: \tau(\rho)B$ by Lemma 3. This implies $0 = [f, \tau(\rho)B] = L_f(\tau(\rho))B + \tau(\rho)[f, B] = L_f(\tau(\rho))B$ and τ is constant by Lemma 3. Therefore $0 \neq g - cf = \tilde{g} \in \mathbf{K}^* \cdot B$, and the assertion is proved. ∎

Now we can finally prove the theorem. Supppose that g^* is analytic in 0, $g^* \notin \mathbf{K}f^*$, and $[f^*, g^*] = 0$. Let $H = \mathrm{Id} + \cdots$ be a formal power series that transforms f^* into (formal) normal form f, and g^* into some formal power series g with $[f, g] = 0$. If f satisfies Condition A, then we know that H can be chosen as a convergent series. If f does not satisfy Condition A then we have $B = c_1^{-1}(g - c_2 f)$ from Lemma 4. Let $\tilde{g}^* := c_1^{-1}(g^* - c_2 f^*)$. Then H transforms $\tilde{g}^*$ into B, and B satisfies Condition A, of course. Therefore there is a convergent $\hat{H} = \mathrm{Id} + \cdots$ transforming $\tilde{g}^*$ into normal form, and this normal form is equal to B by virtue of [2, Theorem 2, p.155]; see also [7, Proposition 1.5]. Define $\hat{f}$ by

$$D\hat{H}(x)\,f^*(x) = \hat{f}(\hat{H}(x)),$$

then $\hat{f}$ is analytic in 0 and $[B, \hat{f}] = 0$ follows from $[\tilde{g}^*, f^*] = 0$, thus $\hat{f}$ is in normal form. This finishes the proof.

3. CONCLUDING REMARKS

While we only considered the case that the centralizer of B (in the Lie algebra of all formal power series) has infinite dimension, the assertion of the theorem remains true in the other case. This was proved by Markhashov [6]. (As was pointed out in [3], there is a problem with the definition of the number l in Markhashov's article and a mistake in the proof of the main theorem, but an analysis of the proof shows that it does work in the case of a finite dimensional centralizer. In the special case $\lambda_1 = -\lambda_2$ the theorem was proved in [3, Lemma 4], using a different approach.)

We have tried to formulate the lemmata in such a way that they are easy to generalize. For instance, a generalization of Lemma 4 can be proved for arbitrary dimension in the case of "simple resonance"; i.e., when B is semi-simple and every normal form with respect to B is of the type

$$f = B + \sum_{l \geq 1} \rho^l(\alpha_l \operatorname{Id} + D_l),$$

where ρ is a homogeneous polynomial, $L_B(\rho) = 0$, $L_{D_l}(\rho) = 0$, and $[B, D_l] = 0$ for all l. (In [1], this is case (b) of the "one-dimensional normal form," cf. Chap. III, Sect. 2.3.) Then, whenever $L_f(\rho) \neq 0$ and g is a formal power series such that $[g, f] = 0$, then $g = cf + D$, where $c \in \mathbf{K}$, D is linear and $[B, D] = 0$. (The proof can be carried over almost verbatim; note that we tacitly used a generalization of Lemma 1 concerning the shape of the normal form.) However, $L_f(\rho) = 0$ is not equivalent to Condition A in dimension >2, so the theorem does not automatically hold in this situation.

Let us consider one example. For

$$f^*(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot x + \sigma(x)x,$$

where σ is an arbitrary homogeneous polynomial of degree $2m$, there is a convergent normalizing transformation, since $g^*(x) := (x_1^2 + x_2^2)^m x$ satisfies $[f^*, g^*] = 0$.

More important, however, seems to be the "geometric" characterization of what guarantees (or obstructs) the existence of a convergent normalizing transformation. One should note the connection to results of Ito [4, 5] who showed that the existence of "sufficiently many" analytic first integrals (and thus, canonical symmetries) for certain Hamiltonian differential equations is equivalent to the existence of a convergent (canonical) normalizing transformation. Quite generally, a convergent transformation into normal form always implies that the centalizer of f^* in the Lie algebra of all analytic vector fields near 0 is "large." (This can be made into a precise statement, cf. [7], and Markhashov [6].) It may be conjectured that this condition (at least under some mild additional assumptions) is generally also sufficient to guarantee a convergent normalizing transformation.

References

1. A. D. Bruno, "Local Methods in Nonlinear Differential Equations," Springer-Verlag, Berlin/Heidelberg/New York, 1989.
2. A. D. (Brjuno) Bruno, Analytical form of differential equations, I, *Trans. Moscow Math. Soc.* **25** (1971), 131–288.
3. A. D. Bruno, Divergence of the real normalizing transformation, Preprint No. 62, Keldysh Institute of Applied Math., 1979 [In Russian]; English translation, *Selecta Math. Soviet.*, in press.
4. H. Ito, Convergence of Birkhoff normal forms for integrable systems, *Comment. Math. Helv.* **64** (1989), 412–461.
5. H. Ito, Integrability of Hamiltonian systems and Birkhoff normal forms in the simple resonance case, *Math. Ann.* **292** (1992), 411–444.
6. L. M. Markhashov, On the reduction of differential equations to the normal form by an analytic transformation, *J. Appl. Math. Mech.* **38**, No. 5 (1974), 788–790.
7. S. Walcher, On differential equations in normal form, *Math. Ann.* **291** (1991), 293–314.